Chapter 1

集合论

Discrete Mathematics

January 3, 2015

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1.1

Contents

1.2

“离散” 的含义?

离散数学 数学的几个分支的总称, 以研究离散量的结构和相互间的关系为主要目标, 其研究对象一般地是有限个或可数无穷个元素. 比如, 集合论、组合学、数论等等.

离散量 与连续量相对, 离散量是指分散开来的、不存在中间值的量. 比如, 开关 $v.s.$ 音量旋钮.

1.3

离散数学的内容

- 集合论 (Set Theory)
- 线性代数 (Linear Algebra)
- 图论 (Graph Theory)
- 数理逻辑 (Mathematical Logic)

- 数论 (Number Theory)
- 组合论 (Combinatorics)
- 概率论 (Probability Theory)

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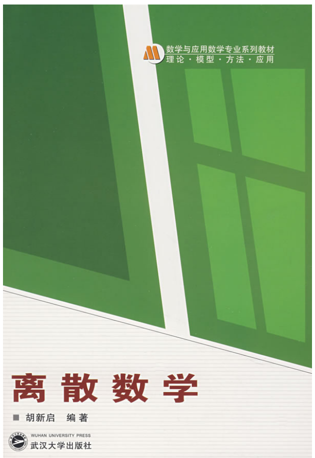
课程教材

References

[1] 胡新启 离散数学 武汉大学出版社, 2007.



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参考书籍

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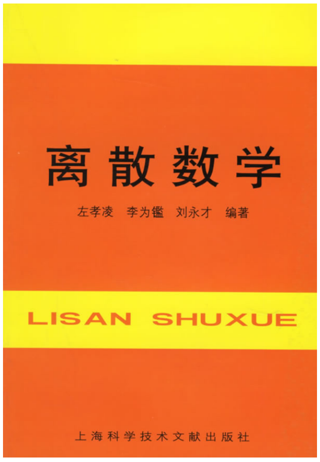
[1] 左孝凌, 李为鑑, 刘永才 离散数学 上海科学技术文献出版社, 1982.



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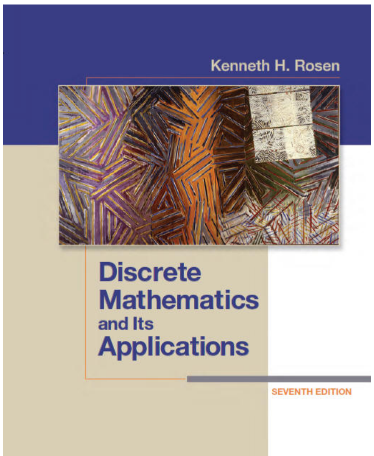

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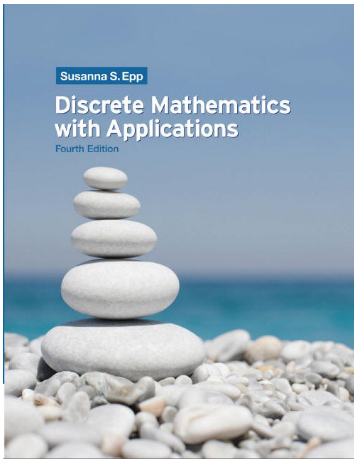

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1 集合的基本概念

集合的定义

- 集合是一个不能精确定义的基本概念.
- 直观地说，集合是具有某种属性的事物的全体，或是一些确定对象的汇合. 而这些事物就是这个集合的元素 (element) 或成员 (member).

- 一个集合把世间万物分成两类, 一些对象属于该集合, 是组成这个集合的成员, 另一些对象不属于该集合.
- 由于一个集合的存在, 世上的对象可分成两类, 任一对象或属于该集合或不属于该集合, 二者必居其一也只居其一.

1.9

集合的记法

- 集合, 通常用大写的英文字母来标记, 如 A, B, C, $\ldots$;
- 元素, 通常用小写字母来表示, 如 a, b, c, $\ldots$.
- 若 a 是集合 A 中的元素, 则称 a 属于集合 A, 记作 $a \in A$.
- 用 $x \notin X$ 表示元素 x 不属于集合 X.

1.10

集合的表示

1. *列举法:* 列出集合的所有元素.
   例如 $A = \{a, b, c, \ldots, z\}$.
2. *描述法:* 用语言概括出集合中元素的特性, 以确定集合的元素.
   例如
   $$B = \{x \mid x \in \mathbb{R} \ \wedge \ x^2 - 1 = 0\}$$
   表示方程 $x^2 - 1 = 0$ 的实数解集.

有的集合可以用两种方法来表示, 如 B 也可以写成 $\{-1, 1\}$. 但是有些集合不能用列举法表示, 如实数集合. 1.11

常见的数集

常见的数的集合, 有约定的记号来表示.

- $\mathbb{N}$ —— 自然数集合 (the set of **natural numbers**. 一般约定 0 是自然数).
  $$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$
- $\mathbb{Z}$ —— 整数集合 (the set of **integers**. Zahl /tsa:l/ *n.*【德】数).
  $$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}.$$
- $\mathbb{Q}$ —— 有理数集合 (the set of **rational numbers**. Quotient, 商).
  $$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z},\ q \in \mathbb{Z},\ \text{and } q \neq 0\}.$$
- $\mathbb{R}$ —— 实数集合 (the set of **real numbers**).
- $\mathbb{C}$ —— 复数集合 (the set of **complex numbers**).
- $\mathbb{Z}^+$ —— 正整数集合.
  $$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}.$$

1.12

集合的性质

集合的基本特性:

- 确定性. 任一元素或属于该集合或不属于该集合, 两者必居其一.
- 无重复性. 集合的元素是彼此不同的. 如果同一个元素在集合中多次出现, 应该认为是一个元素. 如
$$\{1,1,2,2,3\} = \{1,2,3\}.$$
- 无序性. 集合的元素是无序的. 如
$$\{1,2,3\} = \{3,1,2\}.$$
- 抽象性. 集合中的元素是抽象的, 甚至可以是集合. 例如,
$$S = \Big\{a, \{b, c\}, \{\{d\}\}\Big\},$$
其中 a, $\{b,c\}$, $\big\{\{d\}\big\}$ 是集合 S 的元素. 注意 $b \in \{b, c\}$, 但 $b \notin S$, 即 b 并不是集合 S 的元素.

1.13

空集与全集

Definition 1 (空集)**.** 不包含任何元素的集合, 称为空集 (empty set), 记作 $\varnothing$.

例如, $A = \{x \mid x \in \mathbb{R} \wedge x^2 + 1 = 0\}$ 是空集, 式中 $\mathbb{R}$ 表示实数集合.

Definition 2 (全集)**.** 在一定范围内, 如果所有涉及的集合都是某一集合的子集, 则称该集合为全集或论域 (Universe), 一般记作 U 或者 E.

1.14

有限集与无限集

- 只含有有限多个元素的集合称为有限集 (finite sets), 否则称为无限集 (infinite sets).
- 有限集合中元素的个数称为集合的基数 (cardinality).
- 集合 A 的基数表示为 $|A|$, 或者 $\mathrm{card}(A)$. 例如,
$$|\varnothing| = 0.$$

1.15

2 子集与集合的相等

集合间的关系

Theorem 3. *两个集合 A 和 B 相等, 当且仅当它们包含的元素相同.*

集合 A 与 B 相等, 记作 $A = B$. 若 A 与 B 不相等, 则记作 $A \neq B$.

Definition 4 (子集 & 真子集)**.** 如果 A 中每个元素都是 B 中的元素, 则称 A 为 B 的子集 (subset), 记作 $A \subseteq B$ 或 $B \supseteq A$. 如果 $A \subseteq B$ 且 $A \neq B$, 则称 A 为 B 的真子集 (proper subset), 记作 $A \subset B$.

$$A \subseteq B \Leftrightarrow (\forall x)(x \in A \to x \in B), \tag{1}$$
$$A \subset B \Leftrightarrow (\forall x)(x \in A \to x \in B) \wedge (\exists x)(x \in B \wedge x \notin A). \tag{2}$$

1.16

集合间的关系

Theorem 5. 设 A, B 为两个集合. $A = B$ 当且仅当 $A \subseteq B$ 且 $B \subseteq A$. 即

$$A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A). \tag{3}$$

Theorem 6. 对于任意集合 A, $\varnothing \subseteq A$. 即空集是一切集合的子集.

平凡子集 (trivial subset)
任意非空集合 A 至少有两个子集: A 和 $\varnothing$. 称 A 和 $\varnothing$ 是 A 的 平凡子集.

1.17

集合间的关系

Theorem 7. 空集是惟一的.

证: 假设存在空集 $\varnothing_1$ 和 $\varnothing_2$, 由前述定理有

$$\varnothing_1 \subseteq \varnothing_2, \qquad \varnothing_2 \subseteq \varnothing_1.$$

根据集合相等的充要条件, 有

$$\varnothing_1 = \varnothing_2.$$

1.18

Example 8. 判断下列命题是否正确:

1. $\varnothing \subseteq \varnothing$;
2. $\varnothing \in \varnothing$;
3. $\varnothing \subseteq \{\varnothing\}$;
4. $\varnothing \in \{\varnothing\}$.

解:

- 因为空集是任何集合的子集, 所以 (1)、(3) 为真.
- (2) 为假. (因为空集不含任何元素.)
- 这里 $\varnothing$ 是集合 $\{\varnothing\}$ 的元素, 所以 (4) 为真.

1.19

注意事项

- 区分 $\in$ 和 $\subseteq$;
- $\{\varnothing\}$ 是一元集, 而不是空集;

$$\mathrm{card}\big(\{\varnothing\}\big) = 1;$$
$$\mathrm{card}(\varnothing) = 0.$$
$$\{\varnothing\} \neq \varnothing;$$
$$\varnothing \in \{\varnothing\}.$$

1.20

罗素悖论 (Russell's paradox, 1901)

设论域是所有集合的集合, 并定义集合

$$S = \{A \mid A \notin A\}.$$

这样, S 是不以自身为元素的全体集合的集合.

那么 "S" 是不是它自己的元素呢?

- 假设 S 不是自己的元素, 那么 S 满足条件 $A \notin A$, 而该条件定义了集合 S, 所以 $S \in S$.
- 另一方面, 如果 $S \in S$, 那么 S 必须满足定义 S 的条件, 所以 $S \notin S$. 1.21

罗素悖论 (1901)

罗素曾经用*理发师悖论* (Barber paradox) 来解释他的悖论:

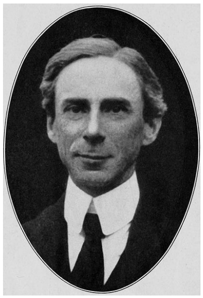
某镇上一位理发师宣布, 他只给那些不给自己刮脸的人刮脸.

那么问题是: *理发师的脸, 谁来刮?!*

罗素悖论起因于集合可以是自己的元素的概念.

康托之后所创立的许多公理化集合论, 都直接或间接地限制集合成为它自己的元素, 从而避免了罗素悖论. 1.22

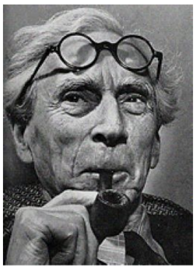
伯特兰・罗素



Bertrand Russell (1872 - 1970) was a British philosopher, logician, mathematician, historian, and social critic. He was born in Wales, into one of the most prominent aristocratic families in Britain. He became an orphan at an early age and was placed in the care of his father's parents, who had him educated at home. He entered Trinity College, Cambridge, in 1890, where he excelled in mathematics and in moral science.

He won a fellowship on the basis of his work on the foundations of geometry. In 1910 Trinity College appointed him to a lectureship in logic and the philosophy of mathematics.

Russell fought for progressive causes throughout his life. He held strong pacifist views, and his protests against World War I led to dismissal from his position at Trinity College. 1.23

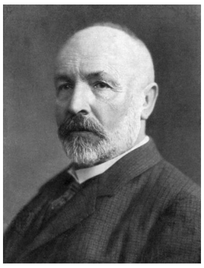


伯特兰·罗素

He was imprisoned for 6 months in 1918 because of an article he wrote that was branded as seditious. Russell fought for women's suffrage in Great Britain. In 1961, at the age of 89, he was imprisoned for the second time for his protests advocating nuclear disarmament.

Russell's greatest work was in his development of principles that could be used as a foundation for all of mathematics. His most famous work is *Principia Mathematica*, written with Alfred North Whitehead, which attempts to deduce all of mathematics using a set of primitive axioms. He wrote many books on philosophy, physics, and his political ideas. Russell won the Nobel Prize for literature in 1950. 1.24

集合论创立者: 康托



Georg Cantor(1845-1918) was born in St. Petersburg, Russia, where his father was a successful merchant. Cantor developed his interest in mathematics in his teens. He began his university studies in Zurich in 1862, but when his father died he left Zurich. He continued his university studies at the University of Berlin in 1863, where he studied under the eminent mathematicians Weierstrass, Kummer, and Kronecker.

He received his doctor's degree in 1867, after having written a dissertation on number theory. Cantor assumed a position at the University of Halle in 1869, where he continued working until his death. Cantor is considered the founder of set theory. His contributions in this area include the discovery that the set of real numbers is uncountable. He is also noted for his many important contributions to analysis. 1.25

集合论创立者: 康托

Cantor also was interested in philosophy and wrote papers relating his theory of sets with metaphysics. Cantor married in 1874 and had five children. His melancholy temperament was balanced by his wife's happy disposition. Although he received a large inheritance from his father, he was poorly paid as a professor. To mitigate this, he tried

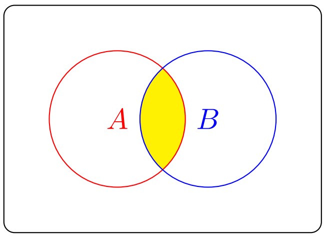


Figure 1: 交集 $A \cap B$.

to obtain a better-paying position at the University of Berlin. His appointment there was blocked by Kronecker, who did not agree with Cantor's views on set theory.

Cantor suffered from mental illness throughout the later years of his life. He died in 1918 in a psychiatric clinic. 1.26

3 集合的运算及其性质

集合的运算

Definition 9 (交集 (intersection of sets))**.** 集合 A 和 B 的交集记为 $A \cap B$, 定义为:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

例如, $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$, 则 $A \cap B = \{1, 3\}$. 进一步有 1.27

$$A_1 \cap A_2 \cap \cdots \cap A_n \triangleq \bigcap_{i=1}^{n} A_i; \tag{4}$$

$$A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots \triangleq \bigcap_{i=1}^{\infty} A_i. \tag{5}$$

1.28

Definition 10 (并集 (union of sets))**.** 集合 A 和 B 的并集记为 $A \cup B$, 定义为:

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

例如, $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$. 则 $A \cup B = \{1, 2, 3, 5\}$.
类似地有, $\bigcup\limits_{i=1}^{n} A_i$, $\bigcup\limits_{i=1}^{\infty} A_i$. 1.29

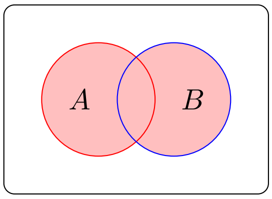


Figure 2: 并集 $A \cup B$.

Definition 11 (差集 (difference set))**.** 集合 A 和 B 的差集 (相对补) 记为 $A - B$(或记为 $A \backslash B$), 定义为:

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$

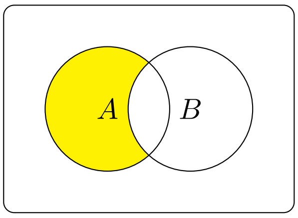


Figure 3: 差集 $A - B$.

例如 $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$. 则 $A - B = \{2\}$. 1.30

Definition 12 (补集 (complement, complementation))**.** 集合 A 的补集 (绝对补) 记为 $\sim A$, 定义为

$$\sim A = E - A = \left\{x \mid x \in E \wedge x \notin A\right\}.$$

常见的记法还有: $\overline{A}$, A^{c}, A'.

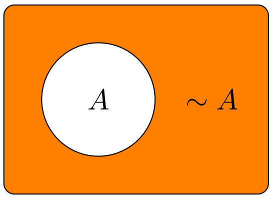


Figure 4: 补集 $\sim A$.

形式上有: $E = \sim A \cup A, \qquad \sim A \cap A = \varnothing.$ 1.31

集合的对称差

Definition 13 (对称差 (symmetric difference))**.** 集合 A 和 B 的对称差 (环和), 记为 $A \oplus B$. 定义为

$$A \oplus B = (A - B) \cup (B - A) = \left\{x \mid x \in A \bar{\vee} x \in B\right\}. \tag{6}$$

或记为 $A + B$. 1.32

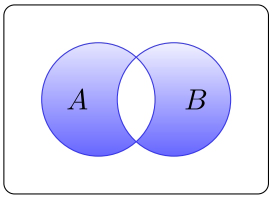


Figure 5: 对称差 $A \oplus B$.

对称差的性质

1. $A \oplus B = (A \cup B) - (A \cap B)$;
2. $B \oplus A = A \oplus B$;
3. $A \oplus A = \varnothing$;
4. $A \oplus \varnothing = A$;
5. $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

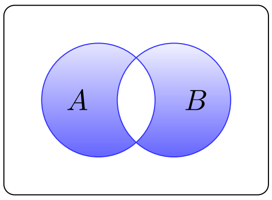


Figure 6: 对称差 $A \oplus B$.

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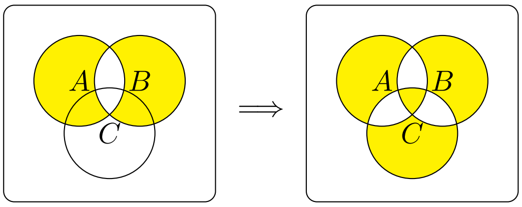
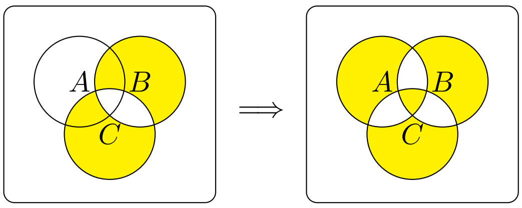
结合律 $(A \oplus B) \oplus C = A \oplus (B \oplus C)$



(a) $(A \oplus B) \oplus C$



(b) $A \oplus (B \oplus C)$

1.34

基本性质

Name	*Identity*
幂等律 (Idempotent laws)	$A \cup A = A, \qquad A \cap A = A$
结合律 (Associative laws)	$(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$
交换律 (Commutative laws)	$A \cup B = B \cup A, \qquad A \cap B = B \cap A$
分配律 (Distributive laws)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
同一律 (Identity laws)	$A \cup \varnothing = A, \qquad A \cap U = A$
零律 (Domination laws)	$A \cup U = U, \qquad A \cap \varnothing = \varnothing$
双补律 (Complement laws)	$A \cup \overline{A} = U, \qquad A \cap \overline{A} = \varnothing$
德・摩根律 (De Morgan's laws)	$\overline{A \cup B} = \overline{A} \cap \overline{B}$, $\overline{A \cap B} = \overline{A} \cup \overline{B}$
吸收律 (Absorption laws)	$A \cup (A \cap B) = A, \qquad A \cap (A \cup B) = A$
对合律 (Complementation law)	$\overline{(\overline{A})} = A$

1.35

Example 14. 确定下列各式:

1. $\varnothing \cap \{\varnothing\}$,
2. $\{\varnothing\} \cap \{\varnothing\}$,
3. $\{\varnothing, \{\varnothing\}\} - \varnothing$,
4. $\{\varnothing, \{\varnothing\}\} - \{\varnothing\}$,
5. $\{\varnothing, \{\varnothing\}\} - \{\{\varnothing\}\}$.

1. $\varnothing \cap \{\varnothing\} = \varnothing$,
2. $\{\varnothing\} \cap \{\varnothing\} = \{\varnothing\}$,
3. $\{\varnothing, \{\varnothing\}\} - \varnothing = \{\varnothing, \{\varnothing\}\}$,
4. $\{\varnothing, \{\varnothing\}\} - \{\varnothing\} = \{\{\varnothing\}\}$,
5. $\{\varnothing, \{\varnothing\}\} - \{\{\varnothing\}\} = \{\varnothing\}$.

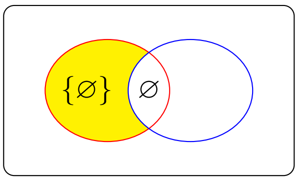


Figure 7: $\{\varnothing, \{\varnothing\}\} - \{\varnothing\}$

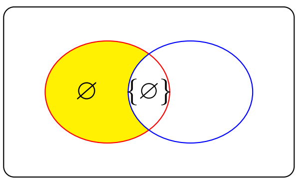


Figure 8: $\{\varnothing, \{\varnothing\}\} - \{\{\varnothing\}\}$

1.36

4 幂集

幂集

Definition 15 (幂集)**.** 给定集合 A, 以 A 的全体子集为元素构成的集合, 称为 A 的幂集 (power set). 记为 $\mathscr{P}(A)$ (或 2^A). 即

$$\mathscr{P}(A) = \{X \mid X \subseteq A\}. \tag{7}$$

Example 16. $A = \{1, 2, 3\}$, 将 A 的子集分类:

- 0 元子集, 也就是空集, 只有一个: $\varnothing$;
- 1 元子集, 即单元集: $\{1\}$, $\{2\}$, $\{3\}$;
- 2 元子集: $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$;
- 3 元子集: $\{1, 2, 3\}$, 即 A 自身.

则 A 的幂集为 $\mathscr{P}(A) = \big\{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\big\}$. 1.37

幂集

Theorem 17. *设 A 为一有限集合, 且 $\mathrm{card}(A) = n$, 那么 $\mathrm{card}\big(\mathscr{P}(A)\big) = 2^n$. 或记为*

$$|\mathscr{P}(A)| = 2^{|A|}.$$

证: 注意到 A 的 0 元子集有 C_n^0 个, A 的 1 元子集有 C_n^1 个, $\cdots$, A 的 k 元子集有 C_n^k 个.

A 的全部子集构成了幂集 $\mathscr{P}(A)$, 所以

$$\begin{aligned} \mathrm{card}\big(\mathscr{P}(A)\big) &= C_n^0 + C_n^1 + \cdots + C_n^k + \cdots + C_n^n \\ &= (1+1)^n \\ &= 2^n. \end{aligned}$$

□ 1.38

幂集

Example 18. 确定下列集合的幂集:

1. $A = \big\{a, \{a\}\big\}$;
2. $\mathscr{P}(\varnothing)$;
3. $\mathscr{P}\big(\mathscr{P}(\varnothing)\big)$.

解: ① $\mathscr{P}(A) = \Big\{\varnothing, \{a\}, \{\{a\}\}, \{a, \{a\}\}\Big\}$.

② 因为 $\mathscr{P}(\varnothing) = \{\varnothing\}$, 所以

$$\mathscr{P}\big(\mathscr{P}(\varnothing)\big) = \big\{\varnothing, \{\varnothing\}\big\}.$$

③ $\mathscr{P}\Big(\mathscr{P}\big(\mathscr{P}(\varnothing)\big)\Big) = \Big\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\Big\}$.

(☞ 注意检验 $\mathrm{card}\big(\mathscr{P}(A)\big) = 2^n$.) 1.39

5 序偶与笛卡儿积

序偶

Definition 19 (序偶)**.** 两个元素 x, y 构成的有序二元组 $\langle x, y\rangle$, 叫序偶 (ordered pairs) 或有序对. 通常把 $\langle x, y\rangle$ 定义为

$$\langle x, y\rangle \triangleq \{\{x\}, \{x, y\}\}.$$

称 x 为 $\langle x, y\rangle$ 的第一元素 (或第一分量、坐标), 称 y 为第二元素 (或第二分量、坐标).

☞ 在集合中 $\{a, b\} = \{b, a\}$, 它们是无序偶; 但对于序偶 $\langle a, b\rangle \neq \langle b, a\rangle$. 1.40

序偶

Definition 20 (序偶的相等)**.** 对任意序偶 $\langle a, b\rangle$, $\langle c, d\rangle$,

$$\langle a, b\rangle = \langle c, d\rangle, \quad \text{当且仅当 } a = c, b = d.$$

1.41

三元序组 (三元组)

Definition 21 (三元序组)**.** 三元序组是一个序偶, 其第一个元素本身也是一个序偶, 可形式化表示为

$$\big\langle\langle x, y\rangle, z\big\rangle \tag{8}$$

约定三元序组可以记作: $\langle x, y, z\rangle$.

☞

$$\big\langle\langle x, y\rangle, z\big\rangle \neq \big\langle x, \langle y, z\rangle\big\rangle.$$

因为 $\big\langle x, \langle y, z\rangle\big\rangle$ 不是三元序组. 1.42

n 元序组 (ordered n-tuples)

Definition 22 (n 元序组)**.** 递归地定义 n 元序组 $\langle a_1, \cdots, a_n\rangle$:

$$\begin{aligned}
\langle a_1, a_2\rangle &\triangleq \big\{\{a_1\}, \{a_1, a_2\}\big\} \\
\langle a_1, a_2, a_3\rangle &\triangleq \big\langle\langle a_1, a_2\rangle, a_3\big\rangle \\
&\cdots \\
\langle a_1, \cdots, a_n\rangle &\triangleq \big\langle\langle a_1, \cdots, a_{n-1}\rangle, a_n\big\rangle
\end{aligned}$$

第 i 个元素 x_i 称作 n 元序组的第 i 个坐标.

☞ 注意: n 元序组是一个二元组, 其中第一个分量是 $n-1$ 元序组. 1.43

n 元序组

Theorem 23. 两个 n 元序组相等

$$\langle a_1, a_2, \cdots, a_n\rangle = \langle b_1, b_2, \cdots, b_n\rangle$$
$$\Leftrightarrow (a_1 = b_1) \wedge (a_2 = b_2) \wedge \cdots \wedge (a_n = b_n).$$

1.44

笛卡尔积 (Cartesian product)

Definition 24 (笛卡尔积)**.** 对任意集合 $A_1, A_2, \cdots, A_n$,

1. $A_1 \times A_2$ 称为集合 A_1, A_2 的笛卡尔积或直积, 定义为 $A_1 \times A_2 = \big\{\langle u, v\rangle \mid u \in A_1 \wedge v \in A_2\big\}$.
2. 递归地定义 $A_1 \times A_2 \times \cdots \times A_n$: $A_1 \times A_2 \times \cdots \times A_n = (A_1 \times A_2 \times \cdots \times A_{n-1}) \times A_n$.
3. $A \times A \times \cdots \times A$ 简记为 A^n.

例如, $A = \{1, 2\}$, $B = \{\alpha, \beta, \gamma\}$, 则

$$A \times B = \big\{\langle 1, \alpha\rangle, \langle 1, \beta\rangle, \langle 1, \gamma\rangle, \langle 2, \alpha\rangle, \langle 2, \beta\rangle, \langle 2, \gamma\rangle\big\}.$$

☞ 若 $A = \varnothing$ 或 $B = \varnothing$, 则

$$A \times B = \varnothing.$$

常见的笛卡尔集:

- 二维平面 $\mathbb{R} \times \mathbb{R}$,
- n 维实数空间 $\mathbb{R}^n$ 等.

1.45

笛卡尔积 (Cartesian product)

The adjective *Cartesian* refers to the French mathematician and philosopher René Descartes (who used the name *Renatus Cartesius* in Latin).

René Descartes (1596 – 1650) has been dubbed the “Father of Modern Philosophy”. Descartes’ influence in mathematics is equally apparent; the Cartesian coordinate system was named after him. He is credited as the father of analytical geometry, the bridge between algebra and geometry, crucial to the discovery of infinitesimal calculus and analysis.

He is perhaps best known for the philosophical statement “Cogito ergo sum” (French: *Je pense, donc je suis*; English: *I think, therefore I am*; or *I am thinking, therefore I exist* or *I do think, therefore I do exist*; Chinese: 我思故我在).

1.46

笛卡尔积 (Cartesian product)

Example 25. 设 $A = \{\text{Ace}, \text{King}, \text{Queen}, \text{Jack}, 10, 9, 8, 7, 6, 5, 4, 3, 2\}$, $B = \{\spadesuit, \heartsuit, \clubsuit, \diamondsuit\}$.
则笛卡尔集 $A \times B$ 是

$$
\begin{aligned}
\{ &\langle \text{Ace}, \spadesuit\rangle, \langle \text{King}, \spadesuit\rangle, \langle \text{Queen}, \spadesuit\rangle, \cdots, \langle 3, \spadesuit\rangle, \langle 2, \spadesuit\rangle,\\
&\langle \text{Ace}, \heartsuit\rangle, \langle \text{King}, \heartsuit\rangle, \langle \text{Queen}, \heartsuit\rangle, \cdots, \langle 3, \heartsuit\rangle, \langle 2, \heartsuit\rangle,\\
&\langle \text{Ace}, \clubsuit\rangle, \langle \text{King}, \clubsuit\rangle, \langle \text{Queen}, \clubsuit\rangle, \cdots, \langle 3, \clubsuit\rangle, \langle 2, \clubsuit\rangle,\\
&\langle \text{Ace}, \diamondsuit\rangle, \langle \text{King}, \diamondsuit\rangle, \langle \text{Queen}, \diamondsuit\rangle, \cdots, \langle 3, \diamondsuit\rangle, \langle 2, \diamondsuit\rangle, \}.
\end{aligned}
$$

1.47

笛卡尔积不满足交换律

- $A \times B \neq B \times A$, 当 $A \neq \varnothing \wedge B \neq \varnothing \wedge A \neq B$ 时.

事实上, 因为

$$A \times B = \{\langle a, b\rangle \mid a \in A \wedge b \in B\}, \tag{9}$$
$$B \times A = \{\langle b, a\rangle \mid b \in B \wedge a \in A\}. \tag{10}$$

而序偶 $\langle a, b\rangle \neq \langle b, a\rangle$, 当 $a \neq b$ 时.

Example 26. 例如 $A = \{1, 2\}$, $B = \{\alpha, \beta\}$, 则

$$
\begin{aligned}
A \times B &= \{\langle 1, \alpha\rangle, \langle 1, \beta\rangle, \langle 2, \alpha\rangle, \langle 2, \beta\rangle\},\\
B \times A &= \{\langle \alpha, 1\rangle, \langle \beta, 1\rangle, \langle \alpha, 2\rangle, \langle \beta, 2\rangle\}.
\end{aligned}
$$

显然 $A \times B \neq B \times A$.

1.48

笛卡尔积不满足结合律

- $(A \times B) \times C \neq A \times (B \times C)$, 当 $A \neq \varnothing \wedge B \neq \varnothing \wedge C \neq \varnothing$ 时

事实上, 因为

$$
\begin{aligned}
(A \times B) \times C &= \Big\{\big\langle \langle a, b\rangle, c\big\rangle \mid \langle a, b\rangle \in A \times B \wedge c \in C\Big\} && (11)\\
&= \Big\{\langle a, b, c\rangle \mid a \in A \wedge b \in B \wedge c \in C\Big\},\\
A \times (B \times C) &= \Big\{\big\langle a, \langle b, c\rangle\big\rangle \mid a \in A \wedge \langle b, c\rangle \in B \times C\Big\}. && (12)
\end{aligned}
$$

而 $\big\langle a, \langle b, c\rangle\big\rangle$ 不是三元组, 所以

$$(A \times B) \times C \neq A \times (B \times C).$$

1.49

笛卡儿积的性质

Theorem 27. *设 A, B, C 为任意集合，$*$ 表示 $\cup, \cap$ 或 $-$ 运算，那么有如下结论：*

1. *笛卡尔积对于并、交、差运算可左分配．即：*

$$A \times (B * C) = (A \times B) * (A \times C). \tag{13}$$

2. *笛卡尔积对于并、交、差运算可右分配．即：*

$$(B * C) \times A = (B \times A) * (C \times A). \tag{14}$$

比如我们来证明

$$(B \cap C) \times A = (B \times A) \cap (C \times A), \tag{15}$$

$$(A - B) \times C = (A \times C) - (B \times C). \tag{16}$$

1.50

Example 28. 证明 $(B \cap C) \times A = (B \times A) \cap (C \times A)$.

证: 在集合 $(B \cap C) \times A$ 中任取 $\langle x, y\rangle$，那么

$$\begin{aligned}
&\langle x, y\rangle \in (B \cap C) \times A \\
\Leftrightarrow\ & x \in (B \cap C) \wedge y \in A \\
\Leftrightarrow\ & (x \in B \wedge x \in C) \wedge y \in A \\
\Leftrightarrow\ & x \in B \wedge y \in A \wedge x \in C \wedge y \in A \\
\Leftrightarrow\ & (x \in B \wedge y \in A) \wedge (x \in C \wedge y \in A) \\
\Leftrightarrow\ & \langle x, y\rangle \in (B \times A) \wedge \langle x, y\rangle \in (C \times A) \\
\Leftrightarrow\ & \langle x, y\rangle \in (B \times A) \cap (C \times A).
\end{aligned}$$

所以 $(B \cap C) \times A = (B \times A) \cap (C \times A)$. □ 1.51

Example 29. 证明 $(A - B) \times C = (A \times C) - (B \times C)$.

证: 在集合 $(A \times C) - (B \times C)$ 中任取 $\langle x, y\rangle$，那么

$$\begin{aligned}
&\langle x, y\rangle \in (A \times C) - (B \times C) \\
\Leftrightarrow\ & \langle x, y\rangle \in A \times C \wedge \neg\langle x, y\rangle \in B \times C \\
\Leftrightarrow\ & (x \in A \wedge y \in C) \wedge \neg(x \in B \wedge y \in C) \\
\Leftrightarrow\ & (x \in A \wedge y \in C) \wedge (\neg x \in B \vee \neg y \in C) \\
\Leftrightarrow\ & (x \in A \wedge y \in C) \wedge (x \notin B \vee y \notin C) \\
\Leftrightarrow\ & (x \in A \wedge y \in C \wedge x \notin B) \vee (x \in A \wedge y \in C \wedge y \notin C) \\
\Leftrightarrow\ & (x \in A \wedge y \in C \wedge x \notin B) \vee \mathbf{F} \\
\Leftrightarrow\ & x \in A \wedge x \notin B \wedge y \in C \\
\Leftrightarrow\ & x \in (A - B) \wedge y \in C \\
\Leftrightarrow\ & \langle x, y\rangle \in (A - B) \times C.
\end{aligned}$$

所以 $(A - B) \times C = (A \times C) - (B \times C)$. □ 1.52

Theorem 30. 若 $C \neq \varnothing$, 则

$$A \subseteq B \Leftrightarrow A \times C \subseteq B \times C \Leftrightarrow C \times A \subseteq C \times B.$$

证: 若 $A \subseteq B$, 任取 $\langle x, y \rangle \in A \times C$, 有

$$\begin{aligned} \langle x, y \rangle \in A \times C &\Rightarrow x \in A \wedge y \in C \\ &\Rightarrow x \in B \wedge y \in C \Rightarrow \langle x, y \rangle \in B \times C. \end{aligned}$$

因此, $A \times C \subseteq B \times C$.

反之, 若 $A \times C \subseteq B \times C$. 取 $y \in C$, 则有

$$\begin{aligned} x \in A &\Rightarrow x \in A \wedge y \in C \Leftrightarrow \langle x, y \rangle \in A \times C \\ &\Rightarrow \langle x, y \rangle \in B \times C \Leftrightarrow x \in B \wedge y \in C \\ &\Leftrightarrow x \in B. \end{aligned}$$

因此, $A \subseteq B$. 类似可证: $A \subseteq B \Leftrightarrow (C \times A \subseteq C \times B)$. □ 1.53

Theorem 31. 设 A, B, C, D 为四个非空集合, 则

$$A \times B \subseteq C \times D \Leftrightarrow A \subseteq C, B \subseteq D.$$

证: 若 $A \times B \subseteq C \times D$, 对任意 $x \in A$ 和 $y \in B$ 有

$$\begin{aligned} (x \in A) \wedge (y \in B) &\Rightarrow \langle x, y \rangle \in A \times B \\ &\Rightarrow \langle x, y \rangle \in C \times D \\ &\Rightarrow (x \in C) \wedge (y \in D). \end{aligned}$$

即 $A \subseteq C$ 且 $B \subseteq D$.

反之, 若 $A \subseteq C$ 且 $B \subseteq D$, 对任意 $x \in A$ 和 $y \in B$ 有

$$\begin{aligned} \langle x, y \rangle \in A \times B &\Leftrightarrow (x \in A) \wedge (y \in B) \\ &\Rightarrow (x \in C) \wedge (y \in D) \\ &\Leftrightarrow \langle x, y \rangle \in C \times D. \end{aligned}$$

因此 $A \times B \subseteq C \times D$. □ 1.54

练习

习题

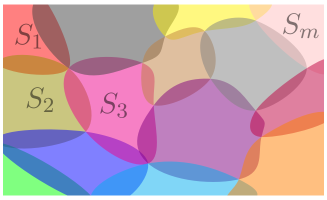
设 $A = \{0, 1\}$, $B = \{1, 2\}$, 确定下面的集合.

1. $A \times \{1\} \times B$;
2. $A^2 \times B$.

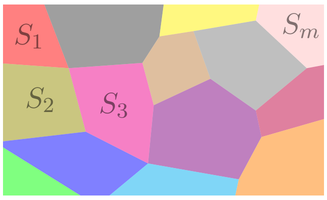
解: ① $A \times \{1\} \times B = \{\langle 0, 1, 1 \rangle, \langle 0, 1, 2 \rangle, \langle 1, 1, 1 \rangle, \langle 1, 1, 2 \rangle\}$.

② $A^2 \times B = A \times A \times B = \{\langle 0, 0, 1 \rangle, \langle 0, 1, 1 \rangle, \langle 0, 0, 2 \rangle, \langle 0, 1, 2 \rangle, \langle 1, 0, 1 \rangle, \langle 1, 0, 2 \rangle, \langle 1, 1, 1 \rangle, \langle 1, 1, 2 \rangle\}$.

1.55



(a) 集合的覆盖



(b) 集合的划分

练习

习题

设 $A = \{a, b\}$, 构成集合 $\mathscr{P}(A) \times A$.

解: 由 $\mathscr{P}(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$, 所以

$$\mathscr{P}(A) \times A = \{\langle\varnothing, a\rangle, \langle\varnothing, b\rangle, \langle\{a\}, a\rangle, \langle\{a\}, b\rangle, \langle\{b\}, a\rangle, \langle\{b\}, b\rangle, \langle\{a, b\}, a\rangle, \langle\{a, b\}, b\rangle\}.$$

1.56

6 集合的覆盖与划分

集合的划分与覆盖

Definition 32 (集合的覆盖 & 划分)**.** 设 A 为非空集合, $S = \{S_1, S_2, \cdots, S_m\}$, 其中 $S_i \subseteq A$, $S_i \neq \varnothing$, $i = 1, 2, \cdots, m$.

- 若 $\bigcup_{i=1}^{m} S_i = A$, 则集合 S 称作集合 A 的覆盖 (covering).
- 若 $\bigcup_{i=1}^{m} S_i = A$, 且 $S_i \cap S_j = \varnothing$ $(i \neq j)$, 则称 S 是 A 的划分 (partition).

1.57

Definition 33 (交叉划分)**.** 若 $\{A_1, A_2, \cdots, A_r\}$ 与 $\{B_1, B_2, \cdots, B_s\}$ 是同一个集合 A 的两种划分, 则其中所有 $A_i \cap B_j \neq \varnothing$ 组成的集合, 称为是原来两种划分的交叉划分.

Example 34. 给定一个玩具积木的集合

$$A = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}.$$

按颜色划分: $\{\{x_1, x_3, x_4, x_7\}, \{x_2, x_5, x_6, x_8\}\}$.

按形状划分: $\{\{x_1, x_2, x_5, x_6\}, \{x_3, x_4, x_7, x_8\}\}$.

同时考虑颜色和形状, 得交叉划分:

$$\{\{x_1\}, \{x_2, x_5, x_6\}, \{x_3, x_4, x_7\}, \{x_8\}.\}.$$

1.58

Theorem 35. 设 $\{A_1, A_2, \cdots, A_r\}$ 与 $\{B_1, B_2, \cdots, B_s\}$ 是同一个集合 X 的两种划分, 则其交叉划分亦是原集合的一种划分.

证: 设 $\{A_1, A_2, \cdots, A_r\}$ 与 $\{B_1, B_2, \cdots, B_s\}$ 交叉划分为

$$T = \left\{ A_i \cap B_j \,\middle|\, 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s \right\}.$$

首先, 在 T 中任取两个元素 $A_i \cap B_h$, $A_j \cap B_k$, 证明它们的交集为空:

1. 当 $i \neq j$, 且 $h = k$ 时,

$$(A_i \cap B_h) \cap (A_j \cap B_k) = (A_i \cap A_j) \cap (B_h \cap B_k) = \varnothing \cap B_h = \varnothing.$$

2. 当 $i \neq j$, 且 $h \neq k$ 时,

$$(A_i \cap B_h) \cap (A_j \cap B_k) = (A_i \cap A_j) \cap (B_h \cap B_k) = \varnothing \cap \varnothing = \varnothing.$$

3. 当 $i = j$, 且 $h \neq k$ 时, 与 ① 类同.

其次, 证明 T 中所有元素的并集等于 X:

$$\begin{aligned}
\bigcup_{i=1}^{r} \bigcup_{h=1}^{s} (A_i \cap B_h) &= \bigcup_{i=1}^{r} (A_i \cap B_1) \cup (A_i \cap B_2) \cup \cdots \cup (A_i \cap B_s) \\
&= \bigcup_{i=1}^{r} \big(A_i \cap (B_1 \cup B_2 \cup \cdots \cup B_s)\big) \\
&= \bigcup_{i=1}^{r} (A_i \cap X) \\
&= \left(\bigcup_{i=1}^{r} A_i \right) \cap X \\
&= X \cap X = X.
\end{aligned}$$

□ 1.59

Definition 36 (划分的加细)**.** 给定 X 的任意两个划分 $\{A_1, A_2, \cdots, A_r\}$ 和 $\{B_1, B_2, \cdots, B_s\}$, 若对于每一个 A_j 均有 B_k 使 $A_j \subseteq B_k$, 则 $\{A_1, A_2, \cdots, A_r\}$ 称为是 $\{B_1, B_2, \cdots, B_s\}$ 的加细.

Theorem 37. 任何两种划分的交叉划分, 都是原来各划分的一种加细.

证: 设 $\{A_1, A_2, \cdots, A_r\}$ 与 $\{B_1, B_2, \cdots, B_s\}$ 交叉划分为

$$T = \left\{ A_i \cap B_j \,\middle|\, 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s \right\}.$$

则对 T 中任意元素 $A_i \cap B_j$, 均有

$$A_i \cap B_j \subseteq A_i, \quad A_i \cap B_j \subseteq B_j.$$

故 T 是原划分的加细. □ 1.60

Example 38. 给定一个玩具积木的集合

$$A = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}.$$

按颜色划分: $\{\{x_1, x_3, x_4, x_7\}, \{x_2, x_5, x_6, x_8\}\}$.

按形状划分: $\{\{x_1, x_2, x_5, x_6\}, \{x_3, x_4, x_7, x_8\}\}$.

同时考虑颜色和形状, 得交叉划分:

$$\{\{x_1\}, \{x_2, x_5, x_6\}, \{x_3, x_4, x_7\}, \{x_8\}\}.$$

从这个例子容易看到:

1. 交叉划分也是原集合的一种划分;
2. 交叉划分是原来各划分的加细. 1.61

7 基本计数原理

鸽巢原理 (pigeonhole principle)

Figure 9: 将 10 只鸽子放进 9 个鸽笼, 那么一定有一个鸽笼放进了至少 2 只鸽子.

Theorem 39 (鸽巢原理). *若有 n 个笼子和 $n+1$ 只鸽子, 所有的鸽子都被关在鸽笼里, 那么至少有一个笼子里有至少 2 只鸽子.* 1.62

鸽巢原理 (pigeonhole principle)

鸽巢原理 (或鸽笼原理), 又名抽屉原理, 最早是由狄利克雷在 1834 年给出的, 常常称为狄利克雷抽屉原理 (Dirichlet's drawer principle), 或简称狄利克雷原理.

鸽巢原理的一个一般表述:

Theorem 40 (鸽巢原理). *将 n 个物体放入 m 个盒子里, 若 $n > m$, 则至少有一个盒子里有两个或两个以上的物体.* 1.63

G. Lejeune Dirichlet

G. Lejeune Dirichlet (1805-1859) was a German mathematician credited with the modern formal definition of a function. He was born into a French family living near Cologne, Germany. He studied at the University of Paris and held positions at the University of Breslau and the University of Berlin. In 1855 he was chosen to succeed Gauss at the University of Göttingen.

Dirichlet made many important discoveries in number theory, including the theorem that there are infinitely many primes in arithmetical progressions $an+b$ when a and b are relatively prime. He proved the $n = 5$ case of Fermat's Last Theorem, that there are no nontrivial solutions in integers to $x^5 + y^5 = z^5$. Dirichlet also made many contributions to analysis. 1.64

鸽巢原理举例

Example 41. 武汉市至少有两个人头发数一样多.

证: 人的头发数一般在 15 万根左右, 可以假定没有人有超过 100 万根头发, 但武汉人口数量为约 700 万.

如果我们让每一个鸽巢对应一个头发数字, 鸽子对应于人, 那就变成了有大于 100 万只鸽子要进到至多 100 万个巢中.

所以. 可以得到 "武汉市至少有两个人头发数一样多" 的结论. 1.65

鸽巢原理举例

Example 42. 有 n 个人互相握手 (不重复握手), 必有两人握手次数相同.

证: 鸽巢对应于握手次数, 鸽子对应于人, 每个人都可以握 $[0,\ n-1]$ 次.

但 0 和 $n-1$ 不能同时存在. 因为如果一个人不和任何人握手, 那就不会存在一个和所有其他人都握过手的人.

所以鸽巢是 $n-1$ 个, 但有 n 个人 (n 只鸽子), 故得证. 1.66

鸽巢原理推广

Theorem 43 (The Generalized Pigeonhole Principle)**.** *把 n 个物体放入 m 个盒子里, 则至少有一个盒子里至少有 $\lceil n/m \rceil$ 个物体.*

☞ 其中 $\lceil x \rceil$ 称为 取顶函数 (ceiling function).

x	$\lfloor x \rfloor$	$\lceil x \rceil$
2.4	2	3
−2.4	−3	−2
−2	−2	−2

与之相对的是 取整函数 (floor function), 记为 $\lfloor x \rfloor$ 或者 $[x]$.

$$\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geqslant x\},$$
$$\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leqslant x\}.$$

例如,

证: 反证法. 设所有的盒子里的物体数都不超过 $\lceil n/m \rceil - 1$, 则物体总数至多为

$$\begin{aligned} & m\left(\left\lceil \frac{n}{m} \right\rceil - 1\right) \\ < {} & m\left(\left(\frac{n}{m} + 1\right) - 1\right) \qquad \left(因为 \left\lceil \tfrac{n}{m} \right\rceil < \tfrac{n}{m} + 1\right) \\ = {} & n. \end{aligned}$$

这与物体总数为 n 矛盾. □

Example 44. 在 100 个人中, 至少有 $\lceil 100/12 \rceil = 9$ 个人的生日, 是在同一个月份. 1.67

鸽巢原理推广

Example 45. 在任意 6 个人中, 或者有 3 个人相互认识, 或者有 3 个人相互陌生.

证: 记这 6 人分别为 A, B, C, D, E, F. 以 A 为例, 把余下 5 人分入 2 个盒子, 其中一个盒子表示和 A 认识, 另一个表示和 A 陌生.

则至少有一个盒子里有至少 $\lceil 5/2 \rceil = 3$ 个人, 即以下情形必居其一:

1. 至少有 3 个人和 A 认识;
2. 至少有 3 个人和 A 陌生.

考虑情形 1. 不妨设 B, C, D 都和 A 认识, 则有以下两种可能:

(i) 若在 B, C, D 这 3 人中, 至少有 2 人相互认识, 则此 6 人中已经有 3 人相互认识.

(ii) 否则 B, C, D 这 3 人相互陌生, 亦有命题成立.

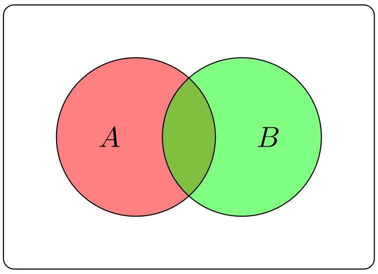
对情形 2 有类似的推理. □ 1.68

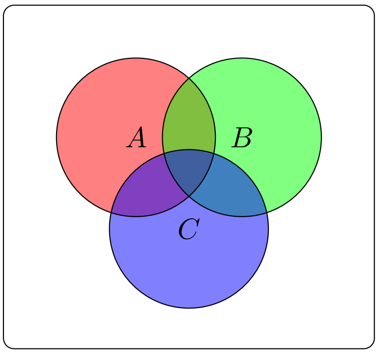
容斥原理 (Inclusion–exclusion principle)

Theorem 46 (容斥原理)**.** 设 A, B 均为有限集, 则有

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

1.69





容斥原理

Theorem 47 (容斥原理)**.** 设 A, B, C 均为有限集, 则有

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

1.70

容斥原理

一般形式:

Theorem 48. 设 A_1, A_2, $\ldots$, A_n $(n \geqslant 2)$ 均为有限集, 则有

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j| + \sum_{1 \leqslant i < j < k \leqslant n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

1.71

概率论中的容斥原理

☞容斥原理在概率论中也有相同的形式:

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2),$$

当 $n = 3$ 时, 公式为:

$$\begin{aligned} \mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) \\ - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \mathbb{P}(A_2 \cap A_3) \\ + \mathbb{P}(A_1 \cap A_2 \cap A_3). \end{aligned}$$

一般地:

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i,j\,:\,i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i,j,k\,:\,i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \ \cdots\ + (-1)^{n-1}\,\mathbb{P}\Big(\bigcap_{i=1}^{n} A_i\Big).$$

1.72

Example 49. A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these people majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Solution: Let A_1 be the set of students who majored in computer science and A_2 the set of students who majored in business. Then $A_1 \cup A_2$ is the set of students who majored in computer science or business (or both), and $A_1 \cap A_2$ is the set of students who majored both in computer science and in business. By the principle of inclusion-exclusion, the number of students who majored either in computer science or in business (or both) equals

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316.$$

We conclude that $350 - 316 = 34$ of the applicants majored neither in computer science nor in business. □ 1.73

Example 50. A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then

$$|S| = 1232, \quad |F| = 879, \quad |R| = 114,$$
$$|S \cap F| = 103, \quad |S \cap R| = 23, \quad |F \cap R| = 14,$$

and

$$|S \cup F \cup R| = 2092.$$

When we insert these quantities into the equation

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|,$$

we obtain

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|.$$

We now solve for $|S \cap F \cap R|$. We find that $|S \cap F \cap R| = 7$. Therefore, there are seven students who have taken courses in all three languages. $\square$ 1.74